

# ALMOST EVERYWHERE STRONG SUMMABILITY OF FEJÉR MEANS OF RECTANGULAR PARTIAL SUMS OF TWO-DIMENSIONAL WALSH-FOURIER SERIES

USHANGI GOGINAVA

**ABSTRACT.** It is proved a BMO-estimation for rectangular partial sums of two-dimensional Walsh-Fourier series from which it is derived an almost everywhere exponential summability of rectangular partial sums of double Walsh-Fourier series.

## 1. INTRODUCTION

We shall denote the set of all non-negative integers by  $\mathbb{N}$ , the set of all integers by  $\mathbb{Z}$  and the set of dyadic rational numbers in the unit interval  $\mathbb{I} := [0, 1)$  by  $\mathbb{Q}$ . In particular, each element of  $\mathbb{Q}$  has the form  $\frac{p}{2^n}$  for some  $p, n \in \mathbb{N}$ ,  $0 \leq p \leq 2^n$ .

Let  $r_0(x)$  be the function defined by

$$r_0(x) = \begin{cases} 1, & \text{if } x \in [0, 1/2) \\ -1, & \text{if } x \in [1/2, 1) \end{cases}, \quad r_0(x+1) = r_0(x).$$

The Rademacher system is defined by

$$r_n(x) = r_0(2^n x), \quad n \geq 1.$$

Let  $w_0, w_1, \dots$  represent the Walsh functions, i.e.  $w_0(x) = 1$  and if  $k = 2^{n_1} + \dots + 2^{n_s}$  is a positive integer with  $n_1 > n_2 > \dots > n_s$  then

$$w_k(x) = r_{n_1}(x) \cdots r_{n_s}(x).$$

The Walsh-Dirichlet kernel is defined by

$$D_0(x) = 0, D_n(x) = \sum_{k=0}^{n-1} w_k(x) \quad n \geq 1.$$

Given  $x \in \mathbb{I}$ , the expansion

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$$(1) \quad x = \sum_{k=0}^{\infty} x_k 2^{-(k+1)},$$

where each  $x_k = 0$  or  $1$ , will be called a dyadic expansion of  $x$ . If  $x \in \mathbb{I} \setminus \mathbb{Q}$ , then (1) is uniquely determined. For the dyadic expansion  $x \in \mathbb{Q}$  we choose the one for which  $\lim_{k \rightarrow \infty} x_k = 0$ .

The dyadic addition of  $x, y \in \mathbb{I}$  in terms of the dyadic expansion of  $x$  and  $y$  is defined by (see [16] or [34])

$$x \dot{+} y = \sum_{k=0}^{\infty} |x_k - y_k| 2^{-(k+1)}.$$

Denote  $I_N := [0, 2^{-N})$ ,  $I_N(x) := x \dot{+} I_N$ .

We consider the double system  $\{w_n(x) \times w_m(y) : n, m \in \mathbb{N}\}$  on the unit square  $\mathbb{I}^2 = [0, 1) \times [0, 1)$ . The notation  $a \lesssim b$  in the whole paper stands for  $a \leq c \cdot b$ , where  $c$  is an absolute constant.

The norm (or quasinorm) of the space  $L_p(\mathbb{I}^2)$  is defined by

$$\|f\|_p := \left( \int_{\mathbb{I}^2} |f|^p \right)^{1/p} \quad (0 < p < +\infty).$$

If  $f \in L_1(\mathbb{I}^2)$ , then

$$\hat{f}(n, m) = \int_{\mathbb{I}^2} f(x_1, x_2) w_n(x_1) w_m(x_2) dx_1 dx_2$$

is the  $(n, m)$ -th Fourier coefficient of  $f$ .

The rectangular partial sums of double Fourier series with respect to the Walsh system are defined by

$$S_{M,N}(x_1, x_2; f) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{f}(m, n) w_m(x_1) w_n(x_2).$$

Denote

$$S_n^{(1)}(x_1, x_2; f) := \sum_{l=0}^{n-1} \hat{f}(l, x_2) w_l(x_1),$$

$$S_m^{(2)}(x_1, x_2; f) := \sum_{r=0}^{m-1} \hat{f}(x_1, r) w_r(x_2),$$

where

$$\hat{f}(l, x_2) = \int_{\mathbb{I}} f(x_1, x_2) w_l(x_1) dx_1, \quad \hat{f}(x_1, r) = \int_{\mathbb{I}} f(x_1, x_2) w_r(x_2) dx_2.$$

Recall the definition of  $BMO [\mathbb{I}^2]$  space. Let  $f \in L_1 (\mathbb{I}^2)$  . We say that  $f$  has bounded mean oscilation ( $f \in BMO [\mathbb{I}^2]$ ) if

$$\|f\|_{BMO} := \sup_Q \left( \frac{1}{|Q|} \int_Q |f - f_Q|^2 \right)^{1/2} < \infty,$$

where

$$f_Q := \frac{1}{|Q|} \int_Q f$$

and the supremum is taken over all dyadic squares  $Q \subset \mathbb{I}^2$ .

Let  $\xi := \{\xi_{n_1 n_2} : n_1, n_2 = 0, 1, 2, \dots\}$  be an arbitrary sequence of numbers. Taking

$$\delta_k^n := \left[ \frac{k}{2^n}, \frac{k+1}{2^n} \right),$$

we define

$$BMO [\xi] := \sup_{0 \leq n_1, n_2 < \infty} \left\| \sum_{k_1=0}^{2^{n_1}-1} \sum_{k_2=0}^{2^{n_2}-1} \xi_{k_1 k_2} \mathbb{I}_{\delta_{k_1}^{n_1}}(t_1) \mathbb{I}_{\delta_{k_2}^{n_2}}(t_2) \right\|_{BMO},$$

where  $\mathbb{I}_E$  is the characteristic function of  $E \subset \mathbb{I}^2$ .

We denote by  $L(\log L)^\alpha (\mathbb{I}^2)$  the class of measurable functions  $f$ , with

$$\int_{\mathbb{I}^2} |f| (\log^+ |f|)^\alpha < \infty,$$

where  $\log^+ u := \mathbb{I}_{(1, \infty)} \log u$ .

Denote by  $S_n^T(x, f)$  the partial sums of the trigonometric Fourier series of  $f$  and let

$$\sigma_n^T(x, f) = \frac{1}{n+1} \sum_{k=0}^n S_k^T(x, f)$$

be the  $(C, 1)$  means. Fejér [1] proved that  $\sigma_n^T(f)$  converges to  $f$  uniformly for any  $2\pi$ -periodic continuous function. Lebesgue in [19] established almost everywhere convergence of  $(C, 1)$  means if  $f \in L_1(\mathbb{T})$ ,  $\mathbb{T} := [-\pi, \pi)$ . The strong summability problem, i.e. the convergence of the strong means

$$(2) \quad \frac{1}{n+1} \sum_{k=0}^n |S_k^T(x, f) - f(x)|^p, \quad x \in \mathbb{T}, \quad p > 0,$$

was first considered by Hardy and Littlewood in [17]. They showed that for any  $f \in L_r(\mathbb{T})$  ( $1 < r < \infty$ ) the strong means tend to 0 a.e., if  $n \rightarrow \infty$ . The Fourier series of  $f \in L_1(\mathbb{T})$  is said to be  $(H, p)$ -summable at  $x \in T$ , if the values (2) converge to 0 as  $n \rightarrow \infty$ . The  $(H, p)$ -summability problem in  $L_1(\mathbb{T})$  has been investigated by Marcinkiewicz [24] for  $p = 2$ , and later by Zygmund [44] for the general case  $1 \leq p < \infty$ . Oskolkov in [26] proved the

following: Let  $f \in L_1(\mathbb{T})$  and let  $\Phi$  be a continuous positive convex function on  $[0, +\infty)$  with  $\Phi(0) = 0$  and

$$\ln \Phi(t) = O(t / \ln \ln t) \quad (t \rightarrow \infty).$$

Then for almost all  $x$

$$(3) \quad \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^n \Phi(|S_k^T(x, f) - f(x)|) = 0.$$

It was noted in [26] that Totik announced the conjecture that (3) holds almost everywhere for any  $f \in L_1(\mathbb{T})$ , provided

$$\ln \Phi(t) = O(t) \quad (t \rightarrow \infty).$$

In [27] Rodin proved

**Theorem R.** *Let  $f \in L_1(\mathbb{T})$ . Then for any  $A > 0$*

$$\lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^n (\exp(A |S_k^T(x, f) - f(x)|) - 1) = 0$$

for a. e.  $x \in \mathbb{T}$ .

Karagulyan [18] proved that the following is true.

**Theorem K.** *Suppose that a continuous increasing function  $\Phi : [0, \infty) \rightarrow [0, \infty)$ ,  $\Phi(0) = 0$ , satisfies the condition*

$$\limsup_{t \rightarrow +\infty} \frac{\log \Phi(t)}{t} = \infty.$$

*Then there exists a function  $f \in L_1(\mathbb{T})$  for which the relation*

$$\limsup_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^n \Phi(|S_k^T(x, f)|) = \infty$$

*holds everywhere on  $\mathbb{T}$ .*

For Walsh system Rodin [29] (see also Schipp [30]) proved that the following is true.

**Theorem R2 (Rodin).** *If  $\Phi(t) : [0, \infty) \rightarrow [0, \infty)$ ,  $\Phi(0) = 0$ , is an increasing continuous function satisfying*

$$(4) \quad \limsup_{t \rightarrow \infty} \frac{\log \Phi(t)}{t} < \infty,$$

*then the partial sums of Walsh-Fourier series of any function  $f \in L_1(\mathbb{I})$  satisfy the condition*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \Phi(|S_k(x; f) - f(x)|) = 0$$

almost everywhere on  $\mathbb{I}$ .

In the paper [6] we established, that, as in trigonometric case [18], the bound (4) is sharp for a.e.  $\Phi$ -summability of Walsh-Fourier series. Moreover, we prove

**Theorem G GK.** *If an increasing function  $\Phi(t) : [0, \infty) \rightarrow [0, \infty)$  satisfies the condition*

$$\limsup_{t \rightarrow \infty} \frac{\log \Phi(t)}{t} = \infty,$$

*then there exists a function  $f \in L_1(\mathbb{I})$  such that*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \Phi(|S_k(x; f)|) = \infty$$

*holds everywhere on  $\mathbb{I}$ .*

The two-dimensional Fejér summability of  $f \in L \log^+ L(\mathbb{I}^2)$  was proved by Zygmund [44] for trigonometric Fourier series and by Móricz, Schipp and Wade [25] (see also Weisz [40]) for Walsh-Fourier series. The two-dimensional strong summability, i. e.

$$\frac{1}{2^{n_1+n_2}} \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} |S_{k_1 k_2}(x_1, x_2; f) - f(x_1, x_2)|^p \rightarrow 0 \quad \text{a. e. as } n \rightarrow \infty$$

was shown by Gogoladze [13] for trigonometric Fourier series and for  $f \in L \log^+ L(\mathbb{T}^2)$ . The same result for more-dimensional Walsh-Fourier series is due to Rodin [28] (see also Weisz [41]). These results show that in the case of two dimensional functions the  $(C; 1, 1)$  summability and  $(C; 1, 1)$  strong summability we have the same maximal convergence spaces. That is, in both cases we have  $L \log^+ L$ .

It is proved in ([12]) a BMO-estimation for quadratic partial sums of two-dimensional trigonometric Fourier series from which it is derived an almost everywhere exponential summability of quadratic partial sums of double Fourier series.

The results on strong summation and approximation of trigonometric Fourier series have been extended for several other orthogonal systems. For instance, concerning the Walsh system see Schipp [31, 32, 33], Fridli and Schipp [2, 3], Leindler [20, 21, 22, 23], Totik [36, 37, 38], Rodin [28], Weisz [41, 42], Gabisonia [4], Goginava, Gogoladze [11].

The problems of summability of multiple Fourier series have been investigated by Gogoladze [14, 15], Wang [39], Zhag [43], Glukhov [7], Goginava [8, 9], Goginava, Gogoladze [10], Gat, Goginava, Karagulyan [5].

## 2. MAIN RESULTS

In this paper we study a BMO-estimation for rectangular partial sums of two-dimensional Walsh-Fourier series.

**Theorem 1.** *If  $f \in L(\log L)^2(\mathbb{I}^2)$ , then*

$$|\{(x_1, x_2) \in \mathbb{I}^2 : BMO[S_{n_1 n_2}(x_1, x_2; f)] > \lambda\}| \lesssim \frac{1}{\lambda} \left( 1 + \int_{\mathbb{I}^2} |f| (\log |f|)^2 \right).$$

The following theorem shows that the rectangular sums of two-dimensional Walsh-Fourier series of a function  $f \in L(\log L)^2(\mathbb{I}^2)$  are almost everywhere exponentially summable to the function  $f$ . It will be obtained from the previous theorem (see [12]) by using the John-Nirenberg theorem.

**Theorem 2.** *Suppose that  $f \in L(\log L)^2(\mathbb{I}^2)$ . Then for any  $A > 0$*

$$\lim_{m_1, m_2 \rightarrow \infty} \frac{1}{2^{m_1+m_2}} \sum_{n_1=0}^{2^{m_1}-1} \sum_{n_2=0}^{2^{m_2}-1} (\exp(A|S_{n_1 n_2}(x_1, x_2; f) - f(x_1, x_2)|) - 1) = 0$$

for a. e.  $(x_1, x_2) \in \mathbb{I}^2$ .

### 3. AUXILIARY RESULTS

Schipp in [30] introduce the following operator

$$V_n(x; f) := \left( \sum_{l=0}^{2^n-1} \left( \int_{l2^{-n}}^{(l+1)2^{-n}} \sum_{j=0}^{n-1} 2^{j-1} \mathbb{I}_{I_j}(t) S_{2^n}(x + t + e_j, f) dt \right)^2 \right)^{1/2},$$

Let

$$V(f) := \sup_n V_n(f).$$

The following theorem is proved by Schipp.

**Theorem Sch1** ([30]). *Let  $f \in L_1(\mathbb{I})$ . Then*

$$\mu\{|Vf| > \lambda\} \lesssim \frac{\|f\|_1}{\lambda}$$

and

$$\|Vf\| \lesssim 1 + \int_{\mathbb{I}} |f| \log^+ |f| \quad (f \in L \log^+ L(\mathbb{I})).$$

**Theorem Sch2** ([30]). *The following estimation holds*

$$\left\{ \frac{1}{2^n} \sum_{m=0}^{2^n-1} |S_m(x; f)|^2 \right\}^{1/2} \lesssim V_n(x; |f|).$$

Set

$$\begin{aligned} & V_{m_1 m_2}(x_1, x_2; f) \\ := & \left( \sum_{l_1=0}^{2^{m_1}-1} \sum_{l_2=0}^{2^{m_2}-1} \left( \int_{l_1 2^{-m_1}}^{(l_1+1)2^{-m_1}} \int_{l_2 2^{-m_2}}^{(l_2+1)2^{-m_2}} \sum_{j_1=0}^{m_1-1} 2^{j_1-1} \mathbb{I}_{I_{j_1}}(t_1) \sum_{j_2=0}^{m_2-1} 2^{j_2-1} \mathbb{I}_{I_{j_2}}(t_2) \right. \right. \end{aligned}$$

$$\times S_{2^{m_1}2^{m_2}}(x_1 \dot{+} t_1 \dot{+} e_{j_1}, x_2 \dot{+} t_2 \dot{+} e_{j_2}; f) dt_1 dt_2)^2)^{1/2}.$$

For a two-dimensional integrable function  $f$  we need to introduce the following functions

$$(5) \quad V_n^{(1)}(x_1, x_2; f) \\ : = \left( \sum_{l=0}^{2^n-1} \left( \int_{l2^{-n}}^{(l+1)2^{-n}} \sum_{j=0}^{n-1} 2^{j-1} \mathbb{I}_{I_j}(t) S_{2^n}^{(1)}(x_1 \dot{+} t \dot{+} e_j, x_2; f)^2 dt \right)^2 \right)^{1/2},$$

$$(6) \quad V_n^{(2)}(x_1, x_2; f) \\ : = \left( \sum_{l=0}^{2^n-1} \left( \int_{l2^{-n}}^{(l+1)2^{-n}} \sum_{j=0}^{n-1} 2^{j-1} \mathbb{I}_{I_j}(t) S_{2^n}^{(2)}(x_1, x_2 \dot{+} t \dot{+} e_j; f) dt \right)^2 \right)^{1/2}, \\ V^{(s)}(x_1, x_2; f) := \sup_n \left| V_n^{(s)}(x_1, x_2; f) \right|, \quad s = 1, 2.$$

**Lemma 1.** *The following estimation holds*

$$\left\{ \frac{1}{2^{m_1+m_2}} \sum_{n_1=0}^{2^{m_1}-1} \sum_{n_2=0}^{2^{m_2}-1} |f * (D_{n_1} \otimes D_{n_2})|^2 \right\}^{1/2} \lesssim V_{m_1 m_2}(x_1, x_2; |f|).$$

*Proof of Lemma 1.* Let

$$\varepsilon_{ji} := \begin{cases} -1, & \text{if } j = 0, 1, \dots, i-1 \\ 1, & \text{if } j = i \end{cases}.$$

In [30], Schipp proved that

$$D_m(t) = \sum_{k=0}^{n-1} \mathbb{I}_{I_k \setminus I_{k+1}}(t) \sum_{j=0}^k \varepsilon_{kj} 2^{j-1} w_m(t + e_j) \\ - \frac{1}{2} w_m(t) + (m+1/2) \mathbb{I}_{I_n}(t), \quad m < 2^n.$$

Then we can write

$$(7) \quad H_{m_1 m_2}(x_1, x_2; f) \\ := \left\{ \frac{1}{2^{m_1+m_2}} \sum_{n_1=0}^{2^{m_1}-1} \sum_{n_2=0}^{2^{m_2}-1} |S_{2^{m_1}2^{m_2}}(f) * (D_{n_1} \otimes D_{n_2})|^2 \right\}^{1/2} \\ \leq \left\{ \frac{1}{2^{m_1+m_2}} \sum_{n_1=0}^{2^{m_1}-1} \sum_{n_2=0}^{2^{m_2}-1} \left| \int_{\mathbb{I}^2} S_{2^{m_1}2^{m_2}}(x_1 \dot{+} t_1, x_2 \dot{+} t_2; f) \sum_{k_1=0}^{m_1-1} \mathbb{I}_{I_{k_1} \setminus I_{k_1+1}}(t_1) \right. \right. \\ \left. \left. \times \sum_{j_1=0}^{k_1} \varepsilon_{k_1 j_1} 2^{j_1-1} w_{n_1}(t_1 \dot{+} e_{j_1}) \sum_{k_2=0}^{m_2-1} \mathbb{I}_{I_{k_2} \setminus I_{k_2+1}}(t_2) \right| \right\}^{1/2}$$

$$\begin{aligned}
& \left. \times \sum_{j_2=0}^{k_2} \varepsilon_{k_2 j_2} 2^{j_2-1} w_{n_2}(t_2 \dot{+} e_{j_2}) dt_1 dt_2 \right|^2 \Bigg\}^{1/2} \\
& + \left\{ \frac{1}{2^{m_1+m_2}} \sum_{n_1=0}^{2^{m_1}-1} \sum_{n_2=0}^{2^{m_2}-1} \left| \int_{\mathbb{I}^2} S_{2^{m_1} 2^{m_2}}(x_1 \dot{+} t_1, x_2 \dot{+} t_2; f) \sum_{k_1=0}^{m_1-1} \mathbb{I}_{I_{k_1} \setminus I_{k_1+1}}(t_1) \right. \right. \\
& \quad \left. \left. \times \sum_{j_1=0}^{k_1} \varepsilon_{k_1 j_1} 2^{j_1-1} w_{n_1}(t_1 \dot{+} e_{j_1}) \frac{w_{n_2}(t_2)}{2} dt_1 dt_2 \right|^2 \right\}^{1/2} \\
& + \left\{ \frac{1}{2^{m_1+m_2}} \sum_{n_1=0}^{2^{m_1}-1} \sum_{n_2=0}^{2^{m_2}-1} \left| \int_{\mathbb{I}^2} S_{2^{m_1} 2^{m_2}}(x_1 \dot{+} t_1, x_2 \dot{+} t_2; f) \sum_{k_1=0}^{m_1-1} \mathbb{I}_{I_{k_1} \setminus I_{k_1+1}}(t_1) \right. \right. \\
& \quad \left. \left. \times \sum_{j_1=0}^{k_1} \varepsilon_{k_1 j_1} 2^{j_1-1} w_{n_1}(t_1 \dot{+} e_{j_1}) (n_2 + 1/2) \mathbb{I}_{I_{m_2}}(t_2) dt_1 dt_2 \right|^2 \right\}^{1/2} \\
& + \left\{ \frac{1}{2^{m_1+m_2}} \sum_{n_1=0}^{2^{m_1}-1} \sum_{n_2=0}^{2^{m_2}-1} \left| \int_{\mathbb{I}^2} S_{2^{m_1} 2^{m_2}}(x_1 \dot{+} t_1, x_2 \dot{+} t_2; f) \sum_{k_2=0}^{m_2-1} \mathbb{I}_{I_{k_2} \setminus I_{k_2+1}}(t_2) \right. \right. \\
& \quad \left. \left. \times \sum_{j_2=0}^{k_2} \varepsilon_{k_2 j_2} 2^{j_2-1} w_{n_2}(t_2 \dot{+} e_{j_2}) \frac{w_{n_1}(t_1)}{2} dt_1 dt_2 \right|^2 \right\}^{1/2} \\
& + \left\{ \frac{1}{2^{m_1+m_2}} \sum_{n_1=0}^{2^{m_1}-1} \sum_{n_2=0}^{2^{m_2}-1} \int_{\mathbb{I}^2} S_{2^{m_1} 2^{m_2}}(x_1 \dot{+} t_1, x_2 \dot{+} t_2; f) \right. \\
& \quad \left. \times \frac{w_{n_1}(t_1)}{2} \frac{w_{n_2}(t_2)}{2} dt_1 dt_2 \right|^2 \Bigg\}^{1/2} \\
& + \left\{ \frac{1}{2^{m_1+m_2}} \sum_{n_1=0}^{2^{m_1}-1} \sum_{n_2=0}^{2^{m_2}-1} \left| \int_{\mathbb{I}^2} S_{2^{m_1} 2^{m_2}}(x_1 \dot{+} t_1, x_2 \dot{+} t_2; f) \right. \right. \\
& \quad \left. \left. \times \frac{w_{n_1}(t_1)}{2} (n_2 + 1/2) \mathbb{I}_{I_{m_2}}(t_2) dt_1 dt_2 \right|^2 \right\}^{1/2} \\
& + \left\{ \frac{1}{2^{m_1+m_2}} \sum_{n_1=0}^{2^{m_1}-1} \sum_{n_2=0}^{2^{m_2}-1} \left| \int_{\mathbb{I}^2} S_{2^{m_1} 2^{m_2}}(x_1 \dot{+} t_1, x_2 \dot{+} t_2; f) \sum_{k_2=0}^{m_2-1} \mathbb{I}_{I_{k_2} \setminus I_{k_2+1}}(t_2) \right. \right. \\
& \quad \left. \left. \times \sum_{j_2=0}^{k_2} \varepsilon_{k_2 j_2} 2^{j_2-1} w_{n_2}(t_2 \dot{+} e_{j_2}) (n_1 + 1/2) \mathbb{I}_{I_{m_1}}(t_1) dt_1 dt_2 \right|^2 \right\}^{1/2}
\end{aligned}$$



$$\begin{aligned}
& + \left\{ \frac{1}{2^{m_1+m_2}} \sum_{n_1=0}^{2^{m_1}-1} \sum_{n_2=0}^{2^{m_2}-1} \left| \int_{\mathbb{I}^2} S_{2^{m_1}2^{m_2}}(x_1+t_1, x_2+t_2; f) \right. \right. \\
& \quad \left. \left. \times \frac{w_{n_2}(t_2)}{2} (n_1+1/2) \mathbb{I}_{I_{m_1}}(t_1) dt_1 dt_2 \right|^2 \right\}^{1/2} \\
& + \left\{ \frac{1}{2^{m_1+m_2}} \sum_{n_1=0}^{2^{m_1}-1} \sum_{n_2=0}^{2^{m_2}-1} \left| \int_{\mathbb{I}^2} S_{2^{m_1}2^{m_2}}(x_1+t_1, x_2+t_2; f) \right. \right. \\
& \quad \left. \left. \times (n_1+1/2) \mathbb{I}_{I_{m_1}}(t_1) (n_2+1/2) \mathbb{I}_{I_{m_2}}(t_2) dt_1 dt_2 \right|^2 \right\}^{1/2} \\
& := \sum_{i=1}^9 R_i.
\end{aligned}$$

There is a suitable vector

$$\left\{ \beta_{n_1 n_2}^{(1)}(x_1, x_2) : 0 \leq n_1 < 2^{m_1}, 0 \leq n_2 < 2^{m_2} \right\}$$

such that

$$\sum_{n_1=0}^{2^{m_1}-1} \sum_{n_2=0}^{2^{m_2}-1} \left| \beta_{n_1 n_2}^{(1)}(x_1, x_2) \right|^2 = 1$$

and

$$\begin{aligned}
(8) \quad & 2^{(m_1+m_2)/2} R_1 \\
& = \int_{\mathbb{I}^2} \sum_{j_1=0}^{m_1-1} 2^{j_1-1} \left( \sum_{k_1=j_1}^{m_1-1} \varepsilon_{k_1 j_1} \mathbb{I}_{I_{k_1} \setminus I_{k_1+1}}(t_1 + e_{j_1}) \right) \\
& \quad \times \sum_{j_2=0}^{m_2-1} 2^{j_2-1} \left( \sum_{k_2=j_2}^{m_2-1} \varepsilon_{k_2 j_2} \mathbb{I}_{I_{k_2} \setminus I_{k_2+1}}(t_2 + e_{j_2}) \right) \\
& \quad \times S_{2^{m_1}2^{m_2}}(x_1+t_1+e_{j_1}, x_2+t_2+e_{j_2}; f) \\
& \times \sum_{n_1=0}^{2^{m_1}-1} \sum_{n_2=0}^{2^{m_2}-1} \beta_{n_1 n_2}^{(1)}(x_1, x_2) w_{n_1}(t_1) w_{n_2}(t_2) dt_1 dt_2 \\
& \leq \int_{\mathbb{I}^2} \sum_{j_1=0}^{m_1-1} 2^{j_1-1} \mathbb{I}_{I_{j_1}}(t_1) \sum_{j_2=0}^{m_2-1} 2^{j_2-1} \mathbb{I}_{I_{j_2}}(t_2) \\
& \quad \times S_{2^{m_1}2^{m_2}}(x_1+t_1+e_{j_1}, x_2+t_2+e_{j_2}; |f|) \\
& \times \left| \sum_{n_1=0}^{2^{m_1}-1} \sum_{n_2=0}^{2^{m_2}-1} \beta_{n_1 n_2}^{(1)}(x_1, x_2) w_{n_1}(t_1) w_{n_2}(t_2) \right| dt_1 dt_2.
\end{aligned}$$

Analogously, we can prove that

$$(9) \quad 2^{(m_1+m_2)/2} R_2 \lesssim \int_{\mathbb{I}^2} \sum_{j_1=0}^{m_1-1} 2^{j_1-1} \mathbb{I}_{I_{j_1}}(t_1) \\ \times S_{2^{m_1} 2^{m_2}}(x_1 \dot{+} t_1 \dot{+} e_{j_1}, x_2 \dot{+} t_2 \dot{+} e_0; |f|) \\ \times \left| \sum_{n_1=0}^{2^{m_1}-1} \sum_{n_2=0}^{2^{m_2}-1} \beta_{n_1 n_2}^{(2)}(x_1, x_2) w_{n_1}(t_1) w_{n_2}(e_0) w_{n_2}(t_2) \right| dt_1 dt_2,$$

$$(10) \quad 2^{(m_1+m_2)/2} R_4 \\ \lesssim \int_{\mathbb{I}^2} \sum_{j_2=0}^{m_2-1} 2^{j_2-1} \mathbb{I}_{I_{j_2}}(t_2) S_{2^{m_1} 2^{m_2}}(x_1 \dot{+} t_1 \dot{+} e_0, x_2 \dot{+} t_2 \dot{+} e_{j_2}; |f|) \\ \times \left| \sum_{n_1=0}^{2^{m_1}-1} \sum_{n_2=0}^{2^{m_2}-1} \beta_{n_1 n_2}^{(4)}(x_1, x_2) w_{n_1}(t_1) w_{n_1}(e_0) w_{n_2}(t_2) \right| dt_1 dt_2.$$

$$(11) \quad 2^{(m_1+m_2)/2} R_5 \\ \lesssim \int_{\mathbb{I}^2} S_{2^{m_1} 2^{m_2}}(x_1 \dot{+} t_1 \dot{+} e_0, x_2 \dot{+} t_2 \dot{+} e_0; |f|) \\ \times \left| \sum_{n_1=0}^{2^{m_1}-1} \sum_{n_2=0}^{2^{m_2}-1} \beta_{n_1 n_2}^{(5)}(x_1, x_2) w_{n_1}(t_1) w_{n_1}(e_0) w_{n_2}(t_2) w_{n_2}(e_0) \right| dt_1 dt_2,$$

where

$$\sum_{n_1=0}^{2^{m_1}-1} \left| \beta_{n_1}^{(s)}(x_1, x_2) \right|^2 = 1 \quad (x_1, x_2) \in \mathbb{I}^2, s = 2, 4, 5.$$

Now, we estimate  $R_3$ . There is a suitable vector

$$\left\{ \beta_{n_1}^{(3)}(x_1, x_2) : 0 \leq n_1 < 2^{m_1} \right\}$$

such that

$$\sum_{n_1=0}^{2^{m_1}-1} \left| \beta_{n_1}^{(3)}(x_1, x_2) \right|^2 = 1, \quad (x_1, x_2) \in \mathbb{I}^2$$

and

$$(12) \quad 2^{(m_1+m_2)/2} R_3 \\ \leq c 2^{3m_2/2} \int_{I \times I_{m_2}} \sum_{j_1=0}^{m_1-1} 2^{j_1-1} \mathbb{I}_{I_{j_1}}(t_1) S_{2^{m_1} 2^{m_2}}(x_1 \dot{+} t_1 \dot{+} e_{j_1}, x_2 \dot{+} t_2; |f|) \\ \times \left| \sum_{n_1=0}^{2^{m_1}-1} \beta_{n_1}^{(3)}(x_1, x_2) w_{n_1}(t_1) \right| dt_1 dt_2.$$

Analogously, we can prove that

$$(13) \quad 2^{(m_1+m_2)/2} R_6 \lesssim 2^{(3/2)m_2} \int_{\mathbb{I} \times I_{m_2}} S_{2^{m_1} 2^{m_2}} (x_1 + t_1 + e_0, x_2 + t_2; |f|) \\ \times \left| \sum_{n_1=0}^{2^{m_1}-1} \beta_{n_1}^{(6)} (x_1, x_2) w_{n_1} (e_0) w_{n_1} (t_1) \right| dt_1 dt_2,$$

$$(14) \quad 2^{(m_1+m_2)/2} R_7 \lesssim 2^{(3/2)m_1} \int_{I_{m_1} \times \mathbb{I}} \sum_{j_2=0}^{m_2-1} 2^{j_2-1} \mathbb{I}_{I_{j_2}} (t_2) \\ \times S_{2^{m_1} 2^{m_2}} (x_1 + t_1, x_2 + t_2 + e_{j_2}; |f|) \\ \times \left| \sum_{n_1=0}^{2^{m_1}-1} \beta_{n_1}^{(7)} (x_1, x_2) w_{n_2} (t_2) \right| dt_1 dt_2,$$

$$(15) \quad 2^{(m_1+m_2)/2} R_8 \lesssim 2^{(3/2)m_1} \int_{I_{m_1} \times \mathbb{I}} S_{2^{m_1} 2^{m_2}} (x_1 \dot{+} t_1, x_2 \dot{+} t_2 \dot{+} e_0; |f|) \\ \times \left| \sum_{n_2=0}^{2^{m_2}-1} \beta_{n_2}^{(8)} (x_1, x_2) w_{n_2} (e_0) w_{n_2} (t_2) \right| dt_1 dt_2,$$

$$(16) \quad 2^{(m_1+m_2)/2} R_9 \lesssim 2^{(3/2)(m_1+m_2)} \int_{I_{m_1} \times I_{m_2}} S_{2^{m_1} 2^{m_2}} (x_1 \dot{+} t_1, x_2 \dot{+} t_2; |f|),$$

where

$$\sum_{n_1=0}^{2^{m_1}-1} \left| \beta_{n_1}^{(s)} (x_1, x_2) \right|^2 = 1 \quad (x_1, x_2) \in \mathbb{I}^2, s = 6, 7$$

and

$$\sum_{n_2=0}^{2^{m_2}-1} \left| \beta_{n_2}^{(8)} (x_1, x_2) \right|^2 = 1 \quad (x_1, x_2) \in \mathbb{I}^2.$$

Set

$$P_{m_1 m_2}^{(1)} (x_1, x_2) := \sum_{n_1=0}^{2^{m_1}-1} \sum_{n_2=0}^{2^{m_2}-1} \beta_{n_1 n_2}^{(1)} (x_1, x_2) w_{n_1} (t_1) w_{n_2} (t_2).$$

Then from (8) we have

$$2^{(m_1+m_2)/2} R_1 \\ = \sum_{l_1=0}^{2^{m_1}-1} \sum_{l_2=0}^{2^{m_2}-1} \left| P_{m_1 m_2}^{(1)} \left( \frac{l_1}{2^{m_1}}, \frac{l_2}{2^{m_2}} \right) \right| \int_{l_1 2^{-m_1}}^{(l_1+1) 2^{-m_1}} \int_{l_2 2^{-m_2}}^{(l_2+1) 2^{-m_2}} \sum_{j_1=0}^{m_1-1} 2^{j_1-1} \mathbb{I}_{I_{j_1}} (t_1)$$

$$\begin{aligned}
& \times \sum_{j_2=0}^{m_2-1} 2^{j_2-1} \mathbb{I}_{I_{j_2}}(t_2) S_{2^{m_1} 2^{m_2}}(x_1 + t_1 + e_{j_1}, x_2 + t_2 + e_{j_2}; |f|) dt_1 dt_2 \\
& \leq \left( \sum_{l_1=0}^{2^{m_1}-1} \sum_{l_2=0}^{2^{m_2}-1} \left| P_{m_1 m_2}^{(1)} \left( \frac{l_1}{2^{m_1}}, \frac{l_2}{2^{m_2}} \right) \right|^2 \right)^{1/2} \\
& \times \left( \sum_{l_1=0}^{2^{m_1}-1} \sum_{l_2=0}^{2^{m_2}-1} \left( \int_{l_1 2^{-m_1}}^{(l_1+1)2^{-m_1}} \int_{l_2 2^{-m_2}}^{(l_2+1)2^{-m_2}} \sum_{j_1=0}^{m_1-1} 2^{j_1-1} \mathbb{I}_{I_{j_1}}(t_1) \sum_{j_2=0}^{m_2-1} 2^{j_2-1} \mathbb{I}_{I_{j_2}}(t_2) \right. \right. \\
& \quad \left. \left. \times S_{2^{m_1} 2^{m_2}}(x_1 + t_1 + e_{j_1}, x_2 + t_2 + e_{j_2}; |f|) dt_1 dt_2 \right)^2 \right)^{1/2}.
\end{aligned}$$

Since

$$\begin{aligned}
& \sum_{l_1=0}^{2^{m_1}-1} \sum_{l_2=0}^{2^{m_2}-1} \left| P_{m_1 m_2}^{(1)} \left( \frac{l_1}{2^{m_1}}, \frac{l_2}{2^{m_2}} \right) \right|^2 \\
& = 2^{m_1+m_2} \sum_{l_1=0}^{2^{m_1}-1} \sum_{l_2=0}^{2^{m_2}-1} \int_{l_1 2^{-m_1}}^{(l_1+1)2^{-m_1}} \int_{l_2 2^{-m_2}}^{(l_2+1)2^{-m_2}} \left| P_{m_1 m_2}^{(1)}(t_1, t_2) \right|^2 dt_1 dt_2 \\
& = 2^{m_1+m_2} \int_{\mathbb{I}^2} \left| P_{m_1 m_2}^{(1)}(t_1, t_2) \right|^2 dt_1 dt_2 \leq 2^{m_1+m_2}
\end{aligned}$$

we have

$$(17) \quad R_1 \lesssim V_{m_1 m_2}(x_1, x_2; |f|)$$

Analogously, from (9)-(16) we can prove that ( $s = 2, \dots, 9$ )

$$(18) \quad R_s \lesssim V_{m_1 m_2}(x_1, x_2; |f|).$$

Combining (7), (17) and (18) we conclude the proof of Theorem 1.  $\square$

**Lemma 2.** *The following estimation holds*

$$V_{m_1 m_2}(x_1, x_2; |f|) \lesssim V^{(1)}(x_1, x_2; V^{(2)}(|f|)).$$

*Proof of Lemma 2.* There is a suitable vector

$$\{a_{l_1 l_2}(x_1, x_2) : 0 \leq l_1 < 2^{m_1}, 0 \leq l_2 < 2^{m_2}\}$$

such that

$$\sum_{l_1=0}^{2^{m_1}-1} \sum_{l_2=0}^{2^{m_2}-1} |a_{l_1 l_2}(x_1, x_2)|^2 = 1$$

and

$$\begin{aligned}
& V_{m_1 m_2}(x_1, x_2; |f|) \\
& = \sum_{l_1=0}^{2^{m_1}-1} \int_{l_1 2^{-m_1}}^{(l_1+1)2^{-m_1}} \sum_{j_1=0}^{m_1-1} 2^{j_1-1} \mathbb{I}_{I_{j_1}}(t_1)
\end{aligned}$$

$$\begin{aligned}
& \times \sum_{l_2=0}^{2^{m_2}-1} a_{l_1 l_2} (x_1, x_2) \left( \int_{l_2 2^{-m_2}}^{(l_2+1)2^{-m_2}} \sum_{j_2=0}^{m_2-1} 2^{j_2-1} \mathbb{I}_{I_{j_2}} (t_2) \right. \\
& \quad \times S_{2^{m_1} 2^{m_2}} (x_1 + t_1 + e_{j_1}, x_2 + t_2 + e_{j_2}; |f|) dt_2 \Big) dt_1 \\
& \leq \sum_{l_1=0}^{2^{m_1}-1} \int_{l_1 2^{-m_1}}^{(l_1+1)2^{-m_1}} \sum_{j_1=0}^{m_1-1} 2^{j_1-1} \mathbb{I}_{I_{j_1}} (t_1) \left( \sum_{l_2=0}^{2^{m_2}-1} |a_{l_1 l_2} (x_1, x_2)|^2 \right)^{1/2} \sum_{l_2=0}^{2^{m_2}-1} \\
& \quad \left( \int_{l_2 2^{-m_2}}^{(l_2+1)2^{-m_2}} \sum_{j_2=0}^{m_2-1} 2^{j_2-1} \mathbb{I}_{I_{j_2}} (t_2) \right. \\
& \quad \times S_{2^{m_2}}^{(2)} (x_1 + t_1 + e_{j_1}, x_2 + t_2 + e_{j_2}; |f|) dt_2 \Big)^{1/2} dt_1 \\
& \leq \sum_{l_1=0}^{2^{m_1}-1} \left( \sum_{l_2=0}^{2^{m_2}-1} |a_{l_1 l_2} (x_1, x_2)|^2 \right)^{1/2} \int_{l_1 2^{-m_1}}^{(l_1+1)2^{-m_1}} \sum_{j_1=0}^{m_1-1} 2^{j_1-1} \mathbb{I}_{I_{j_1}} (t_1) \\
& \quad \times V^{(2)} (x_1 + t_1 + e_{j_1}, x_2; |f|) dt_1 \\
& \leq \left( \sum_{l_1=0}^{2^{m_1}-1} \sum_{l_2=0}^{2^{m_2}-1} |a_{l_1 l_2} (x_1, x_2)|^2 \right)^{1/2} \left( \sum_{l_1=0}^{2^{m_1}-1} \left( \int_{l_1 2^{-m_1}}^{(l_1+1)2^{-m_1}} \sum_{j_1=0}^{m_1-1} 2^{j_1-1} \mathbb{I}_{I_{j_1}} (t_1) \right. \right. \\
& \quad \times V^{(2)} (x_1 + t_1 + e_{j_1}, x_2; |f|) dt_1 \Big)^{1/2} \lesssim V^{(1)} (x_1, x_2; V^{(2)}).
\end{aligned}$$

Lemma 2 is proved.  $\square$

#### 4. PROOF OF MAIN RESULTS

*Proof of Theorem 1.* Set

$$f_{n_1, n_2} (x_1, x_2, t_1, t_2) := \sum_{k_1=0}^{2^{n_1}-1} \sum_{k_2=0}^{2^{n_2}-1} S_{k_1, k_2} (x_1, x_2; f) \mathbb{I}_{\delta_{k_1}^{n_1}} (t_1) \mathbb{I}_{\delta_{k_2}^{n_2}} (t_2),$$

$$J_1 := [j_1 2^{-m}, (j_1 + 1) 2^{-m}), J_2 := [j_2 2^{-m}, (j_2 + 1) 2^{-m}).$$

Then we can write ( $n_1 \leq n_2$ )

$$\begin{aligned}
(19) \quad & \|f_{n_1, n_2} (x_1, x_2, \cdot, \cdot)\|_{BMO} \\
& = \sup_m \sup_{0 \leq j_1, j_2 < 2^m} \left( \frac{1}{|J_1 \times J_2|} \int_{J_1 \times J_2} |f_{n_1, n_2} (x_1, x_2, t_1, t_2)| \right. \\
& \quad \left. - \frac{1}{|J_1 \times J_2|} \int_{J_1 \times J_2} f_{n_1, n_2} (x_1, x_2, u_1, u_2) du_1 du_2 \right)^2 dt_1 dt_2 \Big)^{1/2}
\end{aligned}$$

$$\begin{aligned}
&\leq \left( \sup_{m \leq n_1} \sup_{0 \leq j_1, j_2 < 2^m} + \sup_{n_1 < m \leq n_2} \sup_{0 \leq j_1, j_2 < 2^m} + \sup_{m > n_2} \sup_{0 \leq j_1, j_2 < 2^m} \right) \\
&\quad \left( \frac{1}{|J_1 \times J_2|} \int_{J_1 \times J_2} |f_{n_1, n_2}(x_1, x_2, t_1, t_2)| \right. \\
&\quad \left. - \frac{1}{|J_1 \times J_2|} \int_{J_1 \times J_2} f_{n_1, n_2}(x_1, x_2, u_1, u_2) du_1 du_2 \right)^2 dt_1 dt_2 \Big)^{1/2} \\
&\quad := P_1(n_1, n_2) + P_2(n_1, n_2) + P_3(n_1, n_2).
\end{aligned}$$

et  $n_1 \leq n_2 < m$ . Since  $f_{n_1, n_2}(x_1, x_2, t_1, t_2)$  is constant on  $\left[\frac{j_1}{2^m}, \frac{j_1+1}{2^m}\right) \times \left[\frac{j_2}{2^m}, \frac{j_2+1}{2^m}\right)$  for fixed  $(x_1, x_2) \in \mathbb{I}^2$  we conclude that

$$(20) \quad P_3(n_1, n_2) = 0.$$

Let  $m \leq n_1$ . Then for  $P_1$  we can write

$$\begin{aligned}
&P_1(n_1, n_2) \\
&= \sup_{m \leq n_1} \sup_{0 \leq j_1, j_2 < 2^m} \left( 2^{2m} \int_{J_1 \times J_2} \left| \sum_{k_1=j_1 2^{n_1-m}}^{(j_1+1)2^{n_1-m}-1} \sum_{k_2=j_2 2^{n_2-m}}^{(j_2+1)2^{n_2-m}-1} \right. \right. \\
&\quad \left. S_{k_1, k_2}(x_1, x_2; f) \mathbb{I}_{\delta_{k_1}^{n_1}}(t_1) \mathbb{I}_{\delta_{k_2}^{n_2}}(t_2) - 2^{2m} \right. \\
&\quad \times \int_{J_1 \times J_2} \sum_{k_1=j_1 2^{n_1-m}}^{(j_1+1)2^{n_1-m}-1} \sum_{k_2=j_2 2^{n_2-m}}^{(j_2+1)2^{n_2-m}-1} S_{k_1, k_2}(x_1, x_2; f) \\
&\quad \left. \times \mathbb{I}_{\delta_{k_1}^{n_1}}(u_1) \mathbb{I}_{\delta_{k_2}^{n_2}}(u_2) du_1 du_2 \right|^2 dt_1 dt_2 \Big)^{1/2} \\
&= \sup_{m \leq n_1} \sup_{0 \leq j_1, j_2 < 2^m} \left( 2^{m-n_1} 2^{m-n_2} \sum_{k_1=j_1 2^{n_1-m}}^{(j_1+1)2^{n_1-m}-1} \sum_{k_2=j_2 2^{n_2-m}}^{(j_2+1)2^{n_2-m}-1} \right. \\
&\quad \left. |S_{k_1, k_2}(x_1, x_2; f) - 2^{m-n_1} 2^{m-n_2} \right. \\
&\quad \times \sum_{k_1=j_1 2^{n_1-m}}^{(j_1+1)2^{n_1-m}-1} \sum_{k_2=j_2 2^{n_2-m}}^{(j_2+1)2^{n_2-m}-1} S_{k_1, k_2}(x_1, x_2; f) \Big|^2 \Big)^{1/2} \\
&= \sup_{m_1 \leq n_1, m_2 \leq n_2} \sup_{0 \leq j_1 < 2^{m_1}, 0 \leq j_2 < 2^{m_2}} \left( 2^{-m_1} 2^{-m_2} \sum_{l_1=0}^{2^{m_1}-1} \sum_{l_2=0}^{2^{m_2}-1} \right. \\
&\quad \left. |S_{l_1+j_1 2^{m_1}, l_2+j_2 2^{m_2}}(x_1, x_2; f) - 2^{-m_1} 2^{-m_2} \right.
\end{aligned}$$

$$\times \sum_{q_1=0}^{2^{m_1}-1} \sum_{q_2=0}^{2^{m_2}-1} S_{q_1+j_1 2^{m_1}, q_2+j_2 2^{m_2}}(x_1, x_2; f) \Bigg|^2 \Bigg)^{1/2}.$$

Since

$$\begin{aligned} & S_{l_1+j_1 2^{m_1}, l_2+j_2 2^{m_2}}(x_1, x_2; f) \\ = & S_{j_1 2^{m_1}, j_2 2^{m_2}}(x_1, x_2; f) + S_{l_1, j_2 2^{m_2}}(x_1, x_2; f w_{j_1 2^{m_1}}) w_{j_1 2^{m_1}}(x_1) \\ & + S_{j_1 2^{m_1}, l_2}(x_1, x_2; f w_{j_2 2^{m_2}}) w_{j_2 2^{m_2}}(x_2) \\ & + S_{l_1, l_2}(x_1, x_2; f w_{j_1 2^{m_1}} \otimes w_{j_2 2^{m_2}}) w_{j_1 2^{m_1}}(x_1) w_{j_2 2^{m_2}}(x_2) \end{aligned}$$

we can write

$$\begin{aligned} (21) \quad P_1(n_1, n_2) & \leq \sup_{m_1 \leq n_1, m_2 \leq n_2} \sup_{0 \leq j_1 < 2^{m_1}, 0 \leq j_2 < 2^{m_2}} \left( 2^{-m_1-m_2} \sum_{l_1=0}^{2^{m_1}-1} \sum_{l_2=0}^{2^{m_2}-1} \right. \\ & \quad S_{l_1, l_2}(x_1, x_2; f w_{j_1 2^{m_1}} \otimes w_{j_2 2^{m_2}}) - 2^{-m_1-m_2} \\ & \quad \left. \times \sum_{q_1=0}^{2^{m_1}-1} \sum_{q_2=0}^{2^{m_2}-1} S_{q_1, q_2}(x_1, x_2; f w_{j_1 2^{m_1}} \otimes w_{j_2 2^{m_2}}) \Bigg|^2 \Bigg)^{1/2} \\ & + \sup_{m_1 \leq n_1, m_2 \leq n_2} \sup_{0 \leq j_1 < 2^{m_1}, 0 \leq j_2 < 2^{m_2}} \left( 2^{-m_1} \sum_{l_1=0}^{2^{m_1}-1} |S_{l_1, j_2 2^{m_2}}(x_1, x_2; f w_{j_1 2^{m_1}}) \right. \\ & \quad \left. - 2^{-m_1} \sum_{q_1=0}^{2^{m_1}-1} S_{q_1, j_2 2^{m_2}}(x_1, x_2; f w_{j_1 2^{m_1}}) \Bigg|^2 \Bigg)^{1/2} \\ & + \sup_{m_1 \leq n_1, m_2 \leq n_2} \sup_{0 \leq j_1 < 2^{m_1}, 0 \leq j_2 < 2^{m_2}} \left( 2^{-m_2} \sum_{l_2=0}^{2^{m_2}-1} |S_{j_1 2^{m_1}, l_2}(x_1, x_2; f w_{j_2 2^{m_2}}) \right. \\ & \quad \left. - 2^{-m_2} \sum_{q_2=0}^{2^{m_2}-1} S_{j_1 2^{m_1}, q_2}(x_1, x_2; f w_{j_2 2^{m_2}}) \Bigg|^2 \Bigg)^{1/2} \\ & := P_{11}(n_1, n_2) + P_{12}(n_1, n_2) + P_{13}(n_1, n_2). \end{aligned}$$

From Lemmas 1 and 2 we obtain

$$\begin{aligned} P_{11}(n_1, n_2) & \lesssim \sup_{m_1 \leq n_1, m_2 \leq n_2} \sup_{0 \leq j_1 < 2^{m_1}, 0 \leq j_2 < 2^{m_2}} V_{m_1, m_2}(x_1, x_2; |f w_{j_1 2^{m_1}} \otimes w_{j_2 2^{m_2}}|) \\ & \lesssim V^{(1)}(x_1, x_2; V^{(2)}(|f|)). \end{aligned}$$

Consequently, by Theorem Sch 1 we can write

$$(22) \quad \left| \left\{ (x_1, x_2) \in \mathbb{I}^2 : \sup_{n_1, n_2} P_{11}(n_1, n_2) > \lambda \right\} \right|$$

$$\begin{aligned}
&\lesssim \frac{1}{\lambda} \int_{\mathbb{I}} \left( \int_{\mathbb{I}} V^{(2)}(x_1, x_2; |f|) dx_1 \right) dx_2 \\
&\lesssim \frac{1}{\lambda} \int_{\mathbb{I}} \left( \int_{\mathbb{I}} |f(x_1, x_2)| \log^+ |f(x_1, x_2)| dx_2 + 1 \right) dx_1 \\
&\lesssim \frac{1}{\lambda} \left( \int_{\mathbb{I}^2} |f| \log^+ |f| + 1 \right).
\end{aligned}$$

Since

$$S_{l_1, j_2 2^m}(x_1, x_2; f w_{j_1 2^m}) = S_{l_1}^{(1)}(x_1, x_2; S_{j_2 2^m}(f) w_{j_1 2^m})$$

from Lemma 3 we have

$$\begin{aligned}
&P_{12}(n_1, n_2) \\
&\lesssim \sup_{m_1 \leq n_1, m_2 \leq n_2} \sup_{0 \leq j_1 < 2^{m_1}, 0 \leq j_2 < 2^{m_2}} \left( 2^{-m_1} \sum_{l_1=0}^{2^{m_1}-1} \left| S_{l_1}^{(1)}(x_1, x_2; S_{j_2 2^{m_2}}(f) w_{j_1 2^{m_1}}) \right|^2 \right)^{1/2} \\
&\leq \sup_{m_1 \leq n_1, m_2 \leq n_2} \sup_{0 \leq j_1 < 2^{m_1}, 0 \leq j_2 < 2^{m_2}} V^{(1)}\left(x_1, x_2; \left| S_{j_2 2^{m_2}}^{(2)}(f) w_{j_1 2^{m_1}} \right| \right) \\
&\leq V^{(1)}\left(x_1, x_2; S_*^{(2)}(f)\right),
\end{aligned}$$

where

$$S_*^{(2)}(f) := \sup_n \left| S_n^{(2)}(f) \right|.$$

If  $f \in L(\log^+ L)^2(\mathbb{I}^2)$ . Then  $f(x_1, \cdot) \in L(\log^+ L)^2(\mathbb{I})$  for a. e.  $x_1 \in \mathbb{I}$ , and from the well-known theorem (see [35])  $S_*^{(2)}(x_1, \cdot, f) \in L_1(\mathbb{I})$  for a. e.  $x_1 \in \mathbb{I}$ . Moreover

$$\int_{\mathbb{I}} S_*^{(2)}(x_1, x_2; f) dx_2 \lesssim \int_{\mathbb{I}} |f(x_1, x_2)| (\log^+ |f(x_1, x_2)|)^2 dx_2 + 1$$

for a. e.  $x_1 \in \mathbb{I}$ .

Hence,

$$\begin{aligned}
(23) \quad &\left| \left\{ (x_1, x_2) \in \mathbb{I}^2 : \sup_{n_1, n_2} P_{12}(n_1, n_2) > \lambda \right\} \right| \\
&\lesssim \left| \left\{ (x_1, x_2) \in \mathbb{I}^2 : V^{(1)}\left(x_1, x_2; S_*^{(2)}(f)\right) > \lambda \right\} \right| \\
&\lesssim \frac{1}{\lambda} \int_{\mathbb{I}} \left( \int_{\mathbb{I}} S_*^{(2)}(x_1, x_2; f) dx_1 \right) dx_2 \\
&\lesssim \frac{1}{\lambda} \int_{\mathbb{I}} \left( \int_{\mathbb{I}} |f(x_1, x_2)| (\log^+ |f(x_1, x_2)|)^2 dx_2 + 1 \right) dx_1
\end{aligned}$$



$$\lesssim \frac{1}{\lambda} \left( \int_{\mathbb{I}^2} |f| (\log^+ |f|)^2 + 1 \right).$$

Analogously, we can prove that

$$(24) \quad \left| \left\{ (x_1, x_2) \in \mathbb{I}^2 : \sup_{n_1, n_2} P_{13}(n_1, n_2) > \lambda \right\} \right| \lesssim \frac{1}{\lambda} \left( \int_{\mathbb{I}^2} |f| (\log^+ |f|)^2 + 1 \right).$$

Combining (21)- (24) we get

$$(25) \quad \left| \left\{ (x_1, x_2) \in \mathbb{I}^2 : \sup_{n_1, n_2} P_1(n_1, n_2) > \lambda \right\} \right| \lesssim \frac{1}{\lambda} \left( \int_{\mathbb{I}^2} |f| (\log^+ |f|)^2 + 1 \right).$$

Analogously, we can prove that

$$(26) \quad \left| \left\{ (x_1, x_2) \in \mathbb{I}^2 : \sup_{n_1, n_2} P_2(n_1, n_2) > \lambda \right\} \right| \lesssim \frac{1}{\lambda} \left( \int_{\mathbb{I}^2} |f| (\log^+ |f|)^2 + 1 \right).$$

Combining (19), (20), (25) and (26) we complete the proof of Theorem 1.  $\square$

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U. GOGINAVA, DEPARTMENT OF MATHEMATICS, FACULTY OF EXACT AND NATURAL SCIENCES, IVANE JAVAKHISHVILI TBILISI STATE UNIVERSITY, CHAVCHAVADZE STR. 1, TBILISI 0128, GEORGIA

*E-mail address:* [zazagoginava@gmail.com](mailto:zazagoginava@gmail.com)